



NOVA

University of Newcastle Research Online

nova.newcastle.edu.au

Stoican, Florin, Olaru, Sorin, De Doná, José A., Seron, María M., "Zonotopic ultimate bounds for linear systems with bounded disturbances". Originally published in Proceedings of the 18th World Congress of the International Federation of Automatic Control 2011.

Available from: <http://dx.doi.org/10.3182/20110828-6-IT-1002.03247>

Accessed from: <http://hdl.handle.net/1959.13/1040352>

Zonotopic ultimate bounds for linear systems with bounded disturbances

Florin Stoican* Sorin Olaru* José A. De Doná**
María M. Seron**

* *SUPELEC Systems Sciences (E3S) - Automatic Control Department,
Gif sur Yvette, France*

** *CDSC, School of Electrical Engineering and Computer Science, The
University of Newcastle, NSW 2308, Australia*

Abstract: This paper deals with robust invariant sets construction for discrete-time linear time-invariant dynamics. The case of a zonotopic disturbance set is analysed in detail by exploiting the properties of these geometrical structures. A constructive method is provided for diminishing the conservatism of ultimate bound invariant sets. It is shown that the resulting zonotopic set is related to the minimal robust positively invariant set in the sense that their boundaries have common points.

Keywords: Invariant sets; Zonotopes; Ultimate bounds; Linear Systems.

1. INTRODUCTION

The importance of invariant sets in control has been discussed in, e.g., the popular survey paper Blanchini [1999] and the monography Blanchini and Miani [2007]. Lately, there has been a renewed interest in their characterization [Artstein and Raković, 2008], construction [Raković et al., 2005] and application [Kofman, 2005, Seron et al., 2008]. The present paaper focuses on the class of discrete time-invariant linear systems affected by bounded disturbances. For such systems, the established concepts of minimal robust positively invariant (mRPI) sets and ultimate bound invariant (UBI) sets [Kofman et al., 2007] offer two useful characterizations of invariance. The former are a theoretical tool which may be practically difficult to compute due to the complexity of the representation (in general, a finite characterisation is not assured). On the other hand, UBI sets offer a very direct and simple description but for which one has to accept several sources of conservatism.

The main result of this paper is a description of a UBI set for which the conservatism is greatly reduced by the use of the geometrical properties of polyhedral sets with a specific structure, called *zonotopes*. The two key ideas are that one can find tight zonotopic approximations of the convex disturbance sets first, and then use this zonotope, described as the image of a linear mapping from a higher dimensional hypercube, to obtain a UBI set that preserves the shape of the standard UBI construct (as obtained, e.g., following the techniques in Kofman et al. [2007]) but is squeezed more tightly around the mRPI set.

Notation

Throughout the paper, absolute values and vector inequalities are considered elementwise, that is, $|T|$ denotes the elementwise magnitude of a matrix T and $x \leq y$ ($x < y$) denotes the set of elementwise (strict) inequalities between

the components of the real vectors x and y . For a set $P \in \mathbb{R}^n$ we denote with $\bar{p} = \max_{p \in P} p$ the elementwise

maximum, where each element is computed as $\bar{p}_i = \max_{p \in P} p_i$.

In addition, the elementwise minimum, $\underline{p} = \min_{p \in P} p$, is

defined in a similar way. Given two polyhedral sets P_1, P_2 , the Hausdorff distance is defined as

$$d_H(P_1, P_2) = \max \{ \bar{d}_H(P_1, P_2), \bar{d}_H(P_2, P_1) \}$$

where $\bar{d}_H(P_1, P_2) = \max_{x \in P_1} \min_{y \in P_2} d(x, y)$, and $d(x, y)$ is a

distance measured in a given norm in the $\mathbb{R}^n$ space. For a matrix $A \in \mathbb{R}^{n \times m}$ and a set $P \subseteq \mathbb{R}^m$, we define $AP = \{z \in \mathbb{R}^n : z = Ax \text{ for some } x \in P\}$. The notation B_∞^m represents the ∞ -norm ball in $\mathbb{R}^m$ of radius one. In addition, given a compact set $S \subset \mathbb{R}^m$, $B_\infty^m(S)$ denotes the set of the form $B_\infty^m(S) = \{x : \underline{s} \leq x \leq \bar{s}\}$, where the vector $\underline{s}$, respectively $\bar{s}$, is the elementwise minimum, respectively maximum, of S defined above (note that $B_\infty^m(S)$ is the “smallest box” containing S). e_i denotes the i^{th} standard basis vector.

The rest of the paper is organised as follows: in Section 2 and Section 3 details about zonotopes and invariant sets are presented. The main results are described in Sections 4 and 5 while some conclusions are drawn in Section 6.

2. PRELIMINARY BACKGROUND ON ZONOTOPES

Zonotopes represent a particular class of polytopes characterised by the following definition.

Definition 1. The subset of $\mathbb{R}^n$ with center c and set of generators $\mathcal{G} \triangleq \{g_1, \dots, g_m\} \subset \mathbb{R}^n$, $m \geq n$, such that

$$Z = \left\{ x \in \mathbb{R}^n : x = c + \sum_{i=1}^m \alpha_i g_i, |\alpha_i| \leq 1, g_i \in \mathcal{G} \right\} \quad (1)$$

with $i = 1, \dots, m$ is called a zonotope, denoted in compact form as $Z = (c, < g_1, \dots, g_m >)$. ♦

A zonotope with m generators has the following properties [Fukuda, 2004]:

- is the result of an affine mapping of an m -dimensional hypercube into the $\mathbb{R}^n$ space ($m \geq n$). Thus there exists $C \in \mathbb{R}^{n \times m}$ such that
$$Z = \{c\} \oplus CB_{\infty}^m; \quad (2)$$
- is closed under linear transformation: $LZ = (Lc, <Lg_1, \dots, Lg_m >)$;
- is closed under Minkowski sum: $Z_1 \oplus Z_2 = (c_1 + c_2, <g_1^1, \dots, g_p^1, g_1^2, \dots, g_q^2 >)$.

In this paper, we are interested in more general convex sets, and their zonotopic approximations. Two common cases will be considered:

- Polytopic/polyhedral sets;
- Sets defined by nonlinear inequalities.

In the case of polytopic sets, there are iterative algorithms [see, for example, Dang, 2006] to compute zonotopic approximations in the sense of the following result.

Proposition 1. For any convex and compact polytopic set $\Delta \subset \mathbb{R}^n$ one can construct zonotopes Z_{δ} and $\tilde{Z}_{\delta}$ such that

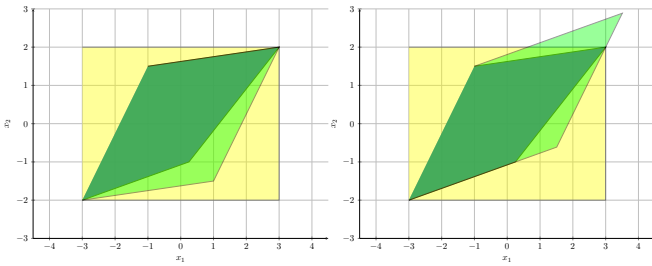
$$\Delta \subseteq Z_{\delta} \subseteq B_{\infty}^n(\Delta), \quad (3)$$

$$d_H(\tilde{Z}_{\delta}, \Delta) \leq d_H(B_{\infty}^n(\Delta), \Delta) \text{ and } \Delta \subseteq \tilde{Z}_{\delta}. \quad (4)$$

□

Proof The result is direct, by observing that $B_{\infty}^n(\Delta)$ is a zonotope such that $Z_{\delta} = \tilde{Z}_{\delta} = B_{\infty}^n(\Delta)$ readily satisfy (3) and (4). Any other zonotope containing Δ is thus a candidate for a tighter approximation. ■

Examples of Proposition 1 are given in Figure 1 where a polytope (with vertices $(0.25, -1)$, $(-1, 1.5)$, $(3, 2)$ and $(-3, -2)$) is approximated by a type (3) zonotope Z_{δ} (with vertices $(1, -1.5)$, $(-1, 1.5)$, $(3, 2)$ and $(-3, -2)$) and by a type (4) zonotope $\tilde{Z}_{\delta}$ (with vertices $(1.5, -0.62)$, $(-3, -2)$, $(-1, 1.5)$ and $(3.5, 2.9)$).



(a) bounding box versus zonotope approximation (b) bounding box versus Hausdorff zonotope $\tilde{Z}_{\delta}$ approximation

Fig. 1. Zonotope bounding of a polytope

A more general case is represented by convex bodies defined by nonlinear inequalities. Common characterisations of such sets include the unit ball of the weighted p -norm (usually some weighted Euclidean norm defining an ellipsoid). In Bourgain and Lindenstrauss [1988] and Linhart [1989] it is proven that any such Euclidean ball can be approximated arbitrarily close, in the sense of the Hausdorff distance, by a zonotope with N generators given by a uniform distribution on the surface of the (hyper)sphere.

In Linhart [1989] one can find lower bounds for N such that the generated zonotope approximates the unit ball within a given Hausdorff distance.

An example is provided in Figure 2 for an ellipsoidal set defined by $x^T P x \leq 1$ with $P = \begin{bmatrix} 1 & 0.1 \\ 0.1 & 2.5 \end{bmatrix}$. Using the results in Linhart [1989] we obtain that for a desired zonotopic approximation of $\epsilon = 0.25$ the required number of generators is $N = 4$.

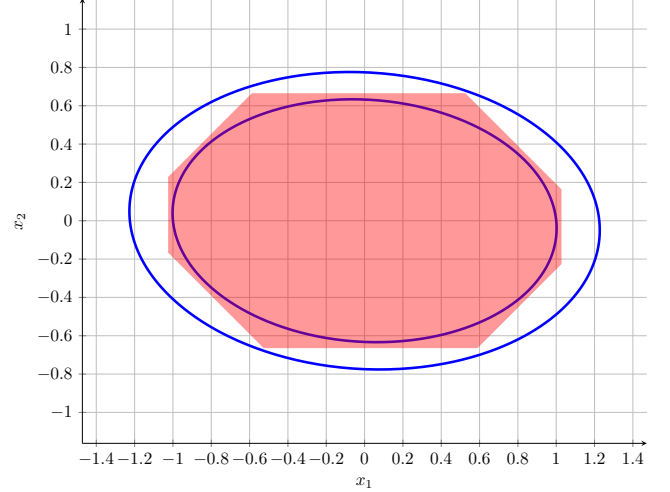


Fig. 2. Zonotope bounding of an ellipsoid

3. INVARIANT SETS

We consider the following discrete-time linear time-invariant dynamic system

$$x^+ = Ax + \delta \quad (5)$$

where $x \in \mathbb{R}^n$ is the current state, $x^+ \in \mathbb{R}^n$ is the successor state and $\delta \in \Delta$ is an unknown disturbance which is assumed bounded (where $\Delta \subset \mathbb{R}^n$ is a convex and compact set). The matrix $A \in \mathbb{R}^{n \times n}$ is assumed strictly stable (i.e., all its eigenvalues are strictly inside the unit circle), diagonalizable and with real eigenvalues.¹

3.1 mRPI set

We consider the following definitions in set invariance analysis (see for example Blanchini [1999]).

Definition 2. The set $\Phi \in \mathbb{R}^n$ is a *robust positively invariant* (RPI) set for (5) if $Ax + \delta \in \Phi$ for all $x \in \Phi$ and for all $\delta \in \Delta$, or, equivalently, if $A\Phi \oplus \Delta \subseteq \Phi$, where $\oplus$ denotes the Minkowski sum of sets. ♦

¹ In an important number of applications, the matrix A in (5) is given by some *closed-loop matrix*, e.g., $\tilde{A} - \tilde{B}K$ or $\tilde{A} - \tilde{L}\tilde{C}$ [where $(\tilde{A}, \tilde{B}, \tilde{C})$ is the open-loop system and K is a feedback gain, L is an observer gain, etc.]. See, e.g., Oлару et al. [2008] and Seron et al. [2008] for some examples of such matrices in connection with the computation of ultimate bound invariant sets. Under standard controllability and observability conditions on $(\tilde{A}, \tilde{B}, \tilde{C})$ the design of K , L , etc., can be readily done by pole placement techniques so that the assumptions on A made here are, without loss of generality, satisfied.

Definition 3. The *minimal robust positively invariant* (mRPI) set Ω_∞ for (5) is the RPI set in $\mathbb{R}^n$ that is contained in every RPI set for (5). ♦

With the given assumptions on matrix A and set Δ , it can be shown that the mRPI set exists, is unique, compact and can be written as $\Omega_\infty = \bigoplus_{i=0}^{\infty} A^i \Delta$. Moreover, it follows from linearity and asymptotic stability of (5) that Ω_∞ is the limit set of all trajectories of (5). Further details can be found in Kolmanovskiy and Gilbert [1998].

3.2 Ultimate bounds

One RPI construction of reduced complexity, and tighter than classical RPI sets rendered by level-set Lyapunov functions, is the one based on ultimate bounds described in Kofman et al. [2007] and applied for different classes of systems in Kofman [2005], Kofman et al. [2008]. The main result for the class of systems we are interested in the current paper, namely (5), is given in the following theorem.

Theorem 2. Consider system (5) and let $A = V\Lambda V^{-1}$ be the Jordan decomposition of matrix A with Λ diagonal and V invertible. Consider also a nonnegative vector $\bar{\delta}$ such that $|\delta| \leq \bar{\delta}, \forall \delta \in \Delta$. Then the set:

$$\Omega_{UB} = \{x \in \mathbb{R}^n : |V^{-1}x| \leq (I - |\Lambda|)^{-1}|V^{-1}|\bar{\delta}\} \quad (6)$$

is robust positively invariant under the dynamics (5). □

Proof: For the proof see Kofman et al. [2007]. ■

3.3 Prolegomenon to zonotopic approximations

In general, it is not possible to compute an exact representation of the mRPI set, except under restrictive assumptions such as when matrix A is nilpotent [Mayne and Schroeder, 1997]. One then needs to resort to approximations, and different algorithms for the construction of RPI approximations can be found in the literature, see for example Raković et al. [2005] and Olaru et al. [2008]. However, those approaches, based on set iterations, focus on the quality of the approximation disregarding its complexity. On the other hand, UBI sets (6) offer an alternative for the construction of RPI sets of low complexity at the price of increased conservativeness. It is worth noticing that the RPI sets that result from the ultimate bound construction (6) (since matrix A has real eigenvalues) are zonotopes. Our goal, in the rest of the paper, is to obtain a tighter zonotopic outer RPI approximation of the mRPI set that preserves the shape and complexity of the UBI set (6).

4. MAIN RESULTS

We consider, without loss of generality, that the set Δ characterizing the disturbance δ in (5) is a zonotope with m generators (if the original set is not a zonotope, an outer zonotopic approximation in the sense of Proposition 1 above can be obtained using available algorithms, e.g., in Dang [2006]). We also consider, without loss of generality, the case where the zonotope Δ is centered at the origin (which is equivalent to $c = 0$ in (2)). Otherwise, a simple change of variables consisting of a translation reduces

the case of a set not centered at the origin to the case considered here.

As explained above (cf. (2)), the zonotopic set Δ centered at the origin can be expressed as an affine mapping of the hypercube in the lifted $\mathbb{R}^m$ space: $\Delta = CB_\infty^m$ with $C \in \mathbb{R}^{n \times m}$, $m \geq n$, a known matrix. Notice, comparing with (1), that the columns of matrix C are the generators of the zonotope Δ (i.e., $C = (g_1, \dots, g_m)$).

We can now state an insightful result with respect to the zonotopic ultimate bounds.

Proposition 3. Consider the zonotopic set of disturbances $\Delta = CB_\infty^m$ and denote with $\bar{w} \in \mathbb{R}^m$ the minimal elementwise positive vector² for which $|w| \leq \bar{w}$ for all $w \in B_\infty^m$. Then, the set

$$\tilde{\Omega}_{UB} = \{x \in \mathbb{R}^n : |V^{-1}x| \leq (I - |\Lambda|)^{-1}|V^{-1}C|\bar{w}\} \quad (7)$$

is a UBI set (which we will call *reduced UBI set*) for system (5) that satisfies the following inclusion:

$$\tilde{\Omega}_{UB} \subseteq \Omega_{UB}. \quad (8)$$

□

Proof: The first part of the proof, namely, that (7) is a UBI set for (5) is evident by noting that system (5) can be written as

$$x^+ = Ax + Cw \quad (9)$$

with $|w| \leq \bar{w}$ and then Theorem 2 in Kofman et al. [2007] can be applied to this system.

Further, since the sets (6) and (7) have the same shape (given by matrix V^{-1}) the verification of inclusion (8) reduces to test that

$$(I - |\Lambda|)^{-1}|V^{-1}C|\bar{w} \leq (I - |\Lambda|)^{-1}|V^{-1}|\bar{\delta} \quad (10)$$

The i^{th} component of $\bar{\delta}$ is given by:

$$\begin{aligned} \bar{\delta}_i &= \max_{\delta \in \Delta} |\delta_i| = \max_{w \in B_\infty^m} |c_i w| \\ &= \max_{w \in B_\infty^m} \left| |c_i| \underbrace{\begin{bmatrix} \ddots & & \\ & \text{sign}(c_{ij}) & \\ & & \ddots \end{bmatrix} w}_{\bar{w}} \right| = \max_{\bar{w} \in B_\infty^m} ||c_i| \bar{w}| \\ &= |c_i| \left[\begin{array}{c} \vdots \\ \max_{\bar{w} \in B_\infty^m} \bar{w}_j \\ \vdots \end{array} \right] = |c_i| \bar{w} \end{aligned}$$

where we denoted with c_i the i^{th} row of C and with c_{ij} the j^{th} element of c_i and used the symmetry of B_∞^m with respect to the origin.

Then $\bar{\delta} = |C|\bar{w}$ and since $|V^{-1}C| \leq |V^{-1}||C|$ it follows that $|V^{-1}C|\bar{w} \leq |V^{-1}|\bar{\delta}$. This is a sufficient condition for verifying (10) since $(I - |\Lambda|)^{-1}$ is a diagonal matrix with

² Note that in the case of B_∞^m the vector $\bar{w}$ is actually

$$\bar{w} = \left[\underbrace{1 \ 1 \ \dots \ 1}_m \right]^T.$$

positive diagonal elements (always the case since matrix A is diagonalisable and stable). ■

In Figure 3 a system with $A = \begin{bmatrix} 0.75 & -0.15 \\ 0.09 & 0.45 \end{bmatrix}$ and generator matrix for the disturbance set Δ , $C = \begin{bmatrix} 3.7 & 8.9 & 2.5 & 1.6 & 3.3 \\ 0.1 & 8.7 & 5.7 & 5.9 & 6.6 \end{bmatrix}$. 10^{-1} is considered in order to illustrate the inclusion and tightness properties (the UBI set, computed as in (6), is represented in blue and the reduced UBI set, computed as in (7), is represented in green).

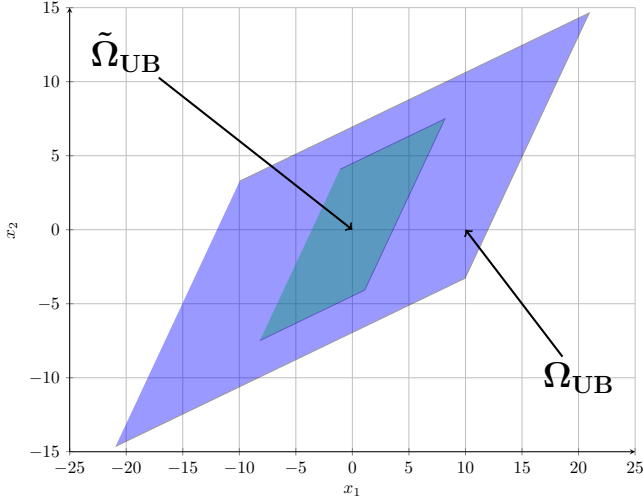


Fig. 3. Example of reduced UBI set $\tilde{\Omega}_{UB}$ versus UBI set Ω_{UB}

4.1 mRPI extreme points

An important property of the reduced UBI set constructed in Proposition 3 is its tightness around the mRPI set associated with system (5):

Theorem 4. Every face of the set $\tilde{\Omega}_{UB}$ is in contact with at least one point of the boundary of the mRPI set Ω_∞ . □

Proof:

In order to prove this result we recall that the mRPI set is a collection of points obtained as infinite sums of all possible combinations of disturbances from the set $\Delta = CB_\infty^m$:

$$\sum_{i=0}^{\infty} A^i \delta_i = \sum_{i=0}^{\infty} A^i C w_i \in \Omega_\infty. \quad (11)$$

We denote a particular subset of points of Ω_∞ , obtained from the infinite series of disturbances $(w, w, w, \dots)$ and ³ $(w, -w, w, \dots)$ acting on system (9) as ⁴:

$$X_w \triangleq \{x_w : x_w = (I \mp A)^{-1} C w, \forall w \in B_\infty^m\}. \quad (12)$$

We can now investigate which of these points, if any, lies on the boundary of $\tilde{\Omega}_{UB}$. Consider the i^{th} equality defining

³ We have the freedom to consider $-w$ for every $w \in B_\infty^m$ as per the symmetry of B_∞^m .

⁴ Note that the convergence of the infinite series is assured by the compactness of the mRPI set, which itself results from the fact that matrix A is strictly stable.

a face of the reduced UBI set $\tilde{\Omega}_{UB}$ and test if there exists a point $x_w \in X_w$ such that

$$e_i^T V^{-1} x_w = e_i^T (I - |\Lambda|)^{-1} |V^{-1} C| \bar{w}. \quad (13)$$

Firstly we compute the left side of (13):

$$\begin{aligned} e_i^T V^{-1} x_w &= e_i^T V^{-1} V (I \mp \Lambda)^{-1} \underbrace{V^{-1} C}_T w \\ &= e_i^T (I \mp \Lambda)^{-1} \sum_j t_j w_j = \frac{1}{1 \mp \lambda_i} e_i^T \sum_j t_j w_j \\ &= \frac{1}{1 \mp \lambda_i} \sum_j t_{ij} w_j \end{aligned} \quad (14)$$

where t_{ij} denotes the (i, j) -th element of matrix $T = V^{-1} C$, t_j denotes the j -th column of matrix T , i.e., $t_j = [t_{1,j} \ t_{2,j} \ \dots \ t_{n,j}]^T$, and where, using the Jordan decomposition $A = V \Lambda V^{-1}$, we have rewritten

$$(I \mp A)^{-1} = (V V^{-1} \mp V \Lambda V^{-1})^{-1} = V (I \mp \Lambda)^{-1} V^{-1}.$$

Applying a similar reasoning to the one used in (14) we obtain the right side of (13) to be (see footnote 2):

$$e_i^T (I - |\Lambda|)^{-1} |V^{-1} C| \bar{w} = \frac{1}{1 - |\lambda_i|} \sum_j |t_{ij}| \quad (15)$$

Using both (14) and (15) we are able to conclude that there exists a point $x_w^{i,+} \in X_w$ that verifies (13):

$$x_w^{i,+} = (I - \text{sign}(\lambda_i) A)^{-1} C \begin{bmatrix} \text{sign}(t_{i1}) \\ \vdots \\ \text{sign}(t_{in}) \end{bmatrix}. \quad (16)$$

The case corresponding to the opposite face of the zonotope $\tilde{\Omega}_{UB}$; that is

$$e_i^T (-V^{-1} x_w) = e_i^T (I - |\Lambda|)^{-1} |V^{-1} C| \bar{w} \quad (17)$$

can be treated analogously, with the point $x_w^{i,-} \in X_w$ verifying condition (17):

$$x_w^{i,-} = -(I - \text{sign}(\lambda_i) A)^{-1} C \begin{bmatrix} \text{sign}(t_{i1}) \\ \vdots \\ \text{sign}(t_{in}) \end{bmatrix}. \quad (18)$$

Gathering all these results we note that the points (16) and (18) lie on the boundary of $\tilde{\Omega}_{UB}$ and at the same time, by construction, reside in the mRPI set Ω_∞ . Hence (cf. Definition 3) these points are also in the boundary of Ω_∞ . This proves that the reduced UBI set is tight, in the sense that it shares boundary points with the boundary of the mRPI set Ω_∞ , thus concluding the proof. ■

Using the same numerical data as in Figure 3, in Figure 4 Ω_∞ (orange), $\tilde{\Omega}_{UB}$ (red), together with several points of X_w are depicted.

Remark 5. Note that the convex hull of the points (16) and (18) will define an inner approximation of the mRPI set Ω_∞ . ♦

The above results were derived under the hypothesis that matrix A is diagonalisable (see footnote 1). Assuming the more general case that A is nondiagonalisable we obtain by means of the Jordan decomposition that matrix Λ will

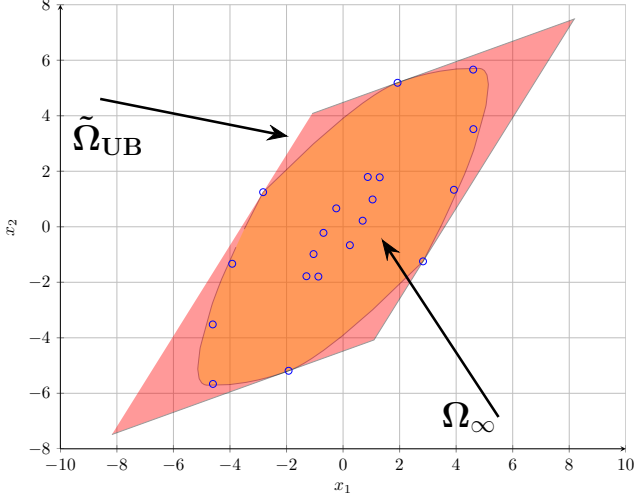


Fig. 4. Example of reduced UBI set $\tilde{\Omega}_{UB}$ and mRPI set Ω_∞ with several points $x_w \in X_w$

be composed of Jordan blocks. Noting that the inverse of a Jordan block associated to an eigenvalue inside the unit circle is a Toeplitz matrix positive elementwise we conclude that matrix $(I - |\Lambda|)^{-1}$ is elementwise positive and upper triangular. We can now retrace the main results of the paper and we remark that Proposition 3 holds while Theorem 4 does not. To see that the first statement is true note that it is sufficient for $(I - |\Lambda|)^{-1}$ to be elementwise positive; as for the second statement, note that the arguments employed in equations (14) to (18) hold only for diagonal matrices.

5. ZONOTOPIC APPROXIMATIONS

As explained in Section 2, if the disturbances are bounded by a polytopic set we aim at obtaining a zonotopic approximation for which there are several alternatives (see, e.g., Proposition 1). It is not *a priori* clear, which of these approximations of the disturbance set will give a better UBI set (7) in the sense of being tight around the mRPI set. The term *better* is itself relative, since various measures can be chosen over $\mathbb{R}^n$ (the most evident being the volume of a set).

Let us consider a polytopic set of disturbances Δ , outer approximated by the members of a collection of zonotopic sets $\{\Delta_i\}_{i=1,\dots,N}$ (e.g. obtained as in Proposition 1):

$$\Delta \subset \Delta_i$$

$$\Delta_i = \left(c_i, \underbrace{\langle g_1^i, g_2^i, \dots, g_{m_i}^i \rangle}_{C_i} \right) \quad (19)$$

For each zonotopic approximation, the dynamics (5) are rewritten by considering the disturbance to be given by the set Δ_i :

$$x^+ = Ax + c_i + C_i w_i, \quad w_i \in B_\infty^{m_i} \quad (20)$$

Through a translation by $(I - A)^{-1}c_i$, the above system is centered at the origin and using Proposition 3 we construct, similarly to (7), a reduced UBI set:

$$\tilde{\Omega}_{UB}^i = \{x \in \mathbb{R}^n : |V^{-1}(x - (I - A)^{-1}c_i)| \leq (I - |\Lambda|)^{-1}|V^{-1}C_i|\bar{w}_i\}. \quad (21)$$

Since we have that $\Delta \subset \Delta_i$ we can conclude that each set (21) constitutes an RPI characterisation for system (5). Consequently, their intersection, $\tilde{\Omega}_{UB}^* = \bigcap_i \tilde{\Omega}_{UB}^i$ can be written as:

$$\tilde{\Omega}_{UB}^* = \left\{ x \in \mathbb{R}^n : \begin{bmatrix} V^{-1} \\ -V^{-1} \end{bmatrix} x \leq \min_i \begin{bmatrix} (I - \Lambda)^{-1}V^{-1}c_i + (I - |\Lambda|)^{-1}|V^{-1}C_i|\bar{w}_i \\ -(I - \Lambda)^{-1}V^{-1}c_i + (I - |\Lambda|)^{-1}|V^{-1}C_i|\bar{w}_i \end{bmatrix} \right\}. \quad (22)$$

Recall that any intersection of RPI sets is also a RPI set. This follows from a simple reasoning: $x \in \bigcap_i \tilde{\Omega}_{UB}^i$

implies that $x \in \tilde{\Omega}_{UB}^i, \forall i$ which by the invariance of each $\tilde{\Omega}_{UB}^i$ means that $x^+ \in \tilde{\Omega}_{UB}^i, \forall i$ which is equivalent to $x^+ \in \bigcap_i \tilde{\Omega}_{UB}^i$. This allows to affirm that the set (22) is also an RPI set for system (5).

As an illustration, consider the system

$$x^+ = \begin{bmatrix} 0.75 & -0.15 \\ 0.09 & 0.45 \end{bmatrix} x + \delta \quad (23)$$

with $\delta \in \Delta$ and $\Delta \subset \mathbb{R}^2$ defined by its set of extreme points $\{(-1, -1), (-0.5, 3), (2, 0.5)\}$.

We consider the three zonotopic approximations $\Delta_{1,2,3}$ depicted in Figure 5(a); where Δ_1 has vertices $(-1, -1), (2, 0.5), (-3.5, 1.5)$ and $(-0.5, 3), \Delta_2$ has vertices $(-1, -1), (2, 0.5), (2.5, 4.5)$ and $(-0.5, 3)$, and Δ_3 has vertices $(-1, -1), (2, 0.5), (-0.5, 3)$ and $(1.5, -3.5)$.

The reduced UBI sets are computed as in (21) (Figure 5(b)) and the RPI set (22) together with the mRPI set associated to system (5) are shown in Figure 5(c).

The tightness of (22), as discussed in Subsection 4.1 can no longer be assured in all cases. This property was studied in Subsection 4.1 with the help of a known set of points (12). The pairs of reduced UBI and mRPI sets associated to each individual system (20) will, for the same reason, share boundary points and each individual approximation can be considered tight. However, since here the zonotopic sets Δ_i are used to *approximate* the *true* polytopic set of disturbances Δ , there is no guarantee that the set (22) will be tight around the mRPI set corresponding to system (5) and disturbance set Δ .

As it can be seen in Figure 5(c) there are cases when the tightness is still verified using the points from (12) (for any hyperplane of the UBI set there exists a shared point with the boundary of Ω_∞). However, changing the matrix A in (23) so that one of its eigenvalues changes sign we observe that we can no longer verify the tightness (as seen in Figure 6 where there are two hyperplanes of the UBI set with no boundary points in the set (12)).

6. CONCLUSIONS

The present paper has dealt with the construction, based on the use of zonotopes, of low complexity robust positively invariant sets. The numerical complexity of the

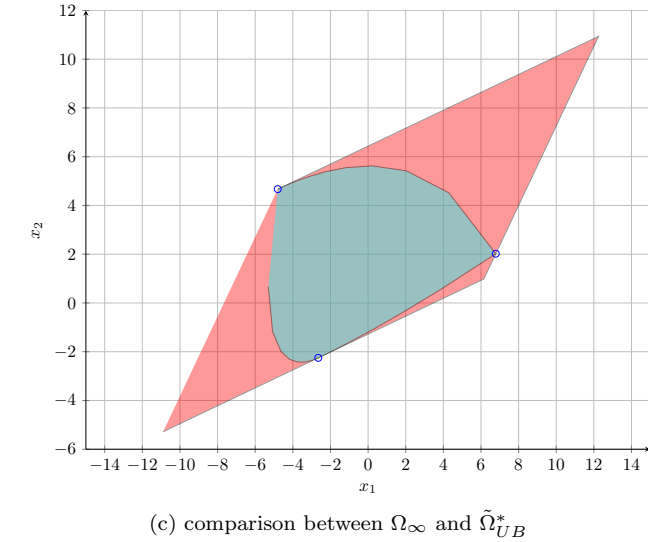
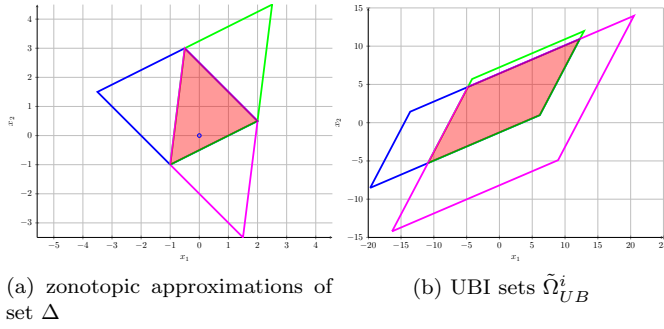


Fig. 5. RPI set obtained from multiple zonotopic approximations and points for testing its tightness

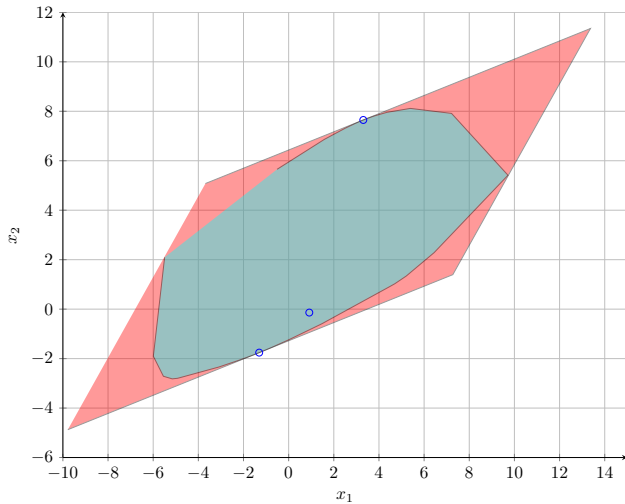


Fig. 6. RPI set obtained from multiple zonotopic approximations and points for testing its tightness

construction amounts to the Jordan decomposition of the system matrix. The ultimate bounds construction was shown to provide a tight approximation of the minimal positive invariant set, in the sense that it shares boundary points with the boundary of the minimal set. Another aspect that has been investigated is the fact that, since any polytope can be expressed as an intersection of zonotopes,

then the resulting tight ultimate bound invariant sets can be intersected so as to reduce the size of the resulting invariant approximation.

REFERENCES

- Zvi Artstein and Sasa V. Raković. Feedback and invariance under uncertainty via set-iterates. *Automatica*, 44(2): 520–525, 2008.
- F. Blanchini. Set invariance in control—a survey. *Automatica*, 35(11):1747–1767, 1999.
- F. Blanchini and S. Miani. *Set-theoretic methods in control*. Birkhauser, 2007.
- J. Bourgain and J. Lindenstrauss. Distribution of points on spheres and approximation by zonotopes. *Israel Journal of Mathematics*, 64(1):25–31, 1988.
- T. Dang. Approximate reachability computation for polynomial systems. *Hybrid Systems: Computation and Control*, pages 138–152, 2006.
- K. Fukuda. From the zonotope construction to the Minkowski addition of convex polytopes. *Journal of Symbolic Computation*, 38(4):1261–1272, 2004.
- Ernesto Kofman. Non-conservative ultimate bound estimation in LTI perturbed systems. *Automatica*, 41(10): 1835–1838, 2005.
- Ernesto Kofman, Hernan Haimovich, and María M. Seron. A systematic method to obtain ultimate bounds for perturbed systems. *International Journal of Control*, 80(2):167–178, 2007.
- Ernesto Kofman, María M. Seron, and Hernan Haimovich. Control design with guaranteed ultimate bound for perturbed systems. *Automatica*, 44(7):1815–1821, 2008.
- I. Kolmanovsky and E.G. Gilbert. Theory and computation of disturbance invariant sets for discrete-time linear systems. *Mathematical Problems in Engineering*, 4:317–367, 1998.
- J. Linhart. Approximation of a ball by zonotopes using uniform distribution on the sphere. *Archiv der Mathematik*, 53(1):82–86, 1989.
- David Q. Mayne and W.R. Schroeder. Robust time-optimal control of constrained linear systems. *Automatica*, 33:2103–2118, 1997.
- Sorin Olaru, José A. De Doná, and María M. Seron. Positive invariant sets for fault tolerant multisensor control schemes. In *Proceedings of the 17th IFAC World Congress*, pages 1224–1229, Seoul, South Korea, 6–11 July 2008.
- Sasa V. Raković, Eric C. Kerrigan, Kostas I. Kouramas, and David Q. Mayne. Invariant approximations of the minimal robust positively invariant set. *IEEE Transactions on Automatic Control*, 50(3):406–410, 2005.
- María M. Seron, Xiang W. Zhuo, José A. De Doná, and J.J. Martinez. Multisensor switching control strategy with fault tolerance guarantees. *Automatica*, 44(1):88–97, 2008. ISSN 0005-1098.